

On the étale site of a marked scheme

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When considering the étale site of a scheme it is often of interest to consider a variant which forces a given set of points to split in at least one member of a covering. Examples are the étale site of a marked curve used in [Scm], where a finite set of closed points is considered and the Nisnevich site [Nis], where all points are required to split. In this note we develop this approach in greater generality.

1 Definition of the marked site

Let X be a scheme and let T be a set of points of X . We will loosely write $T \subset X$ and call the pair (X, T) a *marked scheme*. A morphism $f : (Y, S) \rightarrow (X, T)$ of marked schemes is a scheme morphism $f : Y \rightarrow X$ with $f(S) \subset T$.

Definition 1.1. Let (X, T) be a marked scheme. The *marked étale site* $(X, T)_{\text{et}}$ consists of the following data: The category $\text{Cat}(X, T)_{\text{et}}$ is the category of morphisms $f : (U, S) \rightarrow (X, T)$ such that

- a) $(f : U \rightarrow X)$ is a separated étale morphism of finite presentation, and
- b) $S = p^{-1}(T)$.

A family $(p_i : (U_i, S_i) \rightarrow (U, S))_{i \in I}$ of morphisms in $\text{Cat}(X, T)_{\text{et}}$ is a covering if $(p_i : U_i \rightarrow U)_{i \in I}$ is an étale covering and any point $s \in S$ *splits*, i.e., there exists an index i and a point $u \in S_i$ mapping to s such that the induced field homomorphism $k(s) \rightarrow k(u)$ is an isomorphism.

Example 1.2. For $T = \emptyset$, we obtain the small étale site of X , for $T = X$ the Nisnevich site [Nis].

A morphism of marked schemes induces a morphism of the associated marked étale sites in the obvious way.

The site $(X, T)_{\text{et}}$ has enough points: we fix a separable closure $k(x)^s$ of $k(x)$ for every scheme-theoretic point $x \in X$ and consider the following morphisms of marked schemes

- 1.) for $x \notin T$, the natural morphism $(\text{Spec } k(x)^s, \emptyset) \rightarrow (X, T)$.
- 2.) for $x \in T$, the natural morphisms $(\text{Spec } \kappa, \text{Spec } \kappa) \rightarrow (X, T)$ for every finite subextension $\kappa/k(x)$ of $k(x)^s/k(x)$.

If $f : P \rightarrow X$ is any of the morphisms described in 1.) and 2.), the assignment $F \mapsto \Gamma(P, f^*F)$ is a point of $(X, T)_{\text{et}}$ and one easily verifies that this family of points is conservative. In particular, exactness of sequences of sheaves can be checked stalkwise. We denote the cohomology of a sheaf $F \in \text{Sh}_{\text{et}}(X, T)$ of abelian groups on $(X, T)_{\text{et}}$ by $H_{\text{et}}^*(X, T, F)$.

2 Excision

Let (X, T) be a marked scheme, $i : Z \hookrightarrow X$ a closed immersion and $U = X \setminus Z$ the open complement. The right derivatives of the left exact functor “sections with support in Z ”

$$F \mapsto \ker(F(X, T) \rightarrow F(U, T \cap U))$$

are called the *cohomology groups with support in Z* . Notation: $H_Z^*(X, T, F)$.

The proof of the next proposition is standard (cf. [Art, III, (2.11)] for the étale case without marking).

Proposition 2.1. *There is a long exact sequence*

$$\begin{aligned} 0 \rightarrow H_Z^0(X, T, F) \rightarrow H_{\text{et}}^0(X, T, F) \rightarrow H_{\text{et}}^0(U, T \cap U, F) \rightarrow \\ H_Z^1(X, T, F) \rightarrow H_{\text{et}}^1(X, T, F) \rightarrow H_{\text{et}}^1(U, T \cap U, F) \rightarrow \dots \end{aligned}$$

Proposition 2.2 (Excision). *Let $\pi : (X', T') \rightarrow (X, T)$ be a morphism of marked schemes, $Z \hookrightarrow X$, $Z' \hookrightarrow X'$ closed immersions and $U = X \setminus Z$, $U' = X' \setminus Z'$ the open complements. Assume that*

- $\pi : X' \rightarrow X$ is étale,
- $T' = \pi^{-1}(T)$,
- π induces an isomorphism $Z'_{\text{red}} \xrightarrow{\sim} Z_{\text{red}}$,
- $\pi(U') \subset U$.

Then the induced homomorphism

$$H_Z^p(X, T, F) \xrightarrow{\sim} H_{Z'}^p(X', T', \pi^* F)$$

is an isomorphism for every sheaf $F \in Sh_{\text{et}}(X, T)$ and all $p \geq 0$.

Proof. The standard proof for the étale topology applies: By the general theory, π^* is exact. Since π belongs to $\text{Cat}(X, T)_{\text{et}}$, π^* has the exact left adjoint “extension by zero”, hence π^* sends injectives to injectives. Therefore it suffices to deal with the case $p = 0$. Without changing the statement, we can replace all occurring schemes by their reductions. By assumption,

$$(X', T') \sqcup (U, T \cap U) \rightarrow (X, T)$$

is a covering. For $\alpha \in H_Z^0(X, T, F)$ mapping to zero in $H_{Z'}^0(X', T', \pi^* F)$ we therefore obtain $\alpha = 0$.

Now let $\alpha' \in H_{Z'}^0(X', T', \pi^* F)$ be given. We show that α' and $0 \in H^0(U, T \cap U, F)$ glue to an element in $H_Z^0(X, T, F)$. The only nontrivial compatibility on intersections is $p_1^*(\alpha') = p_2^*(\alpha')$ for $p_1, p_2 : (X' \times_X X', T' \times_T T') \rightarrow (X', T')$. This can be checked on stalks noting that $Z' \xrightarrow{\sim} Z$ implies that the two projections $Z' \times_Z Z' \rightarrow Z'$ are the same. \square

3 Continuity

Theorem 3.1. *Let (X, T) be a marked scheme with T closed in X and let $X_i \rightarrow X$, $i \in I$, be an inverse system of X -schemes over a filtered index category I . Assume that all X_i are quasi-separated and quasi-compact and that all transition morphisms are affine. Let T_i be the preimage of T in X_i and put $X_\infty = \varprojlim X_i$, $T_\infty = \varprojlim T_i$. For a sheaf F of abelian groups on $(X, T)_{\text{et}}$ let F_i and F_∞ denote its inverse image on $(X_i, T_i)_{\text{et}}$ and $(X_\infty, T_\infty)_{\text{et}}$. Then the natural map*

$$\varinjlim_i H_{\text{et}}^p(X_i, T_i, F_i) \longrightarrow H_{\text{et}}^p(X_\infty, T_\infty, F_\infty)$$

is an isomorphism for all $p \geq 0$.

Proof. By the usual trick [SP, Tag 0032], we may assume that I is a directed set. Since X and hence every object of $\text{Cat}(X, T)_{\text{et}}$ is quasi-compact, we may work with the variant of the site which allows only finite covering families. By [Art,

[Thm. III, 3.8], the site $(X_\infty)_{\text{et}}$ is naturally equivalent to the limit site of the $(X_i)_{\text{et}}$. Therefore, after replacing X by an étale, separated X -scheme of finite presentation, it suffices to show that for every quasi-compact étale surjection $U_i \rightarrow X_i$ with the property that every point of T_∞ splits in $U_\infty = U_i \times_{X_i} X_\infty \rightarrow X_\infty$ there exist $j \geq i$ such that every point of T_j splits in $U_j = U_i \times_{X_i} X_j \rightarrow X_j$. We follow the proof of [Scr, Lemma 13.2] for Nisnevich coverings. By [Scr, Lemma 13.3], the subset $F_j \subset T_j$ of points that split in $U_j \rightarrow X_j$ is ind-constructible for all $j \geq i$. Denoting the projection by $u_j : T_\infty \rightarrow T_j$, the assumption on $U_\infty \rightarrow X_\infty$ implies $T_\infty = \bigcup_j u_j^{-1}(F_j)$. Considering the $T_j \subset X_j$ as reduced, closed subschemes, we may apply [EGA4, Cor. 8.3.4] to obtain $F_j = T_j$ for some j . \square

Corollary 3.2. *Let (X, T) be a marked scheme with T closed in X and $Z = \{z_1, \dots, z_n\}$ a finite set of closed points of X . Put $X_{z_i}^h = \text{Spec}(\mathcal{O}_{X, z_i}^h)$. Then, for every sheaf F of abelian groups on (X, T) and all $p \geq 0$*

$$H_Z^p(X, T, F) \cong \bigoplus_{i=1}^n H_{\{z_i\}}^p(X_{z_i}^h, T \cap X_{z_i}^h, F).$$

Proof. Since $H_Z^p(X, T, F) \cong \bigoplus_{i=1}^n H_{\{z_i\}}^p(X, T, F)$, we may assume that $Z = \{z\}$ consists of a single closed point. Excision shows that

$$H_{\{z\}}^p(X, T, F) = H_{\{z\}}^p(U, T \cap U, F)$$

for every affine étale open neighbourhood U of z . Since X_z^h is the limit over all these U , the long exact sequences of proposition 2.1 together with theorem 3.1 show the result. \square

Using theorem 3.1, it is easy to calculate the stalks of the higher direct images of the site morphism $(X, T)_{\text{et}} \rightarrow (X, X)_{\text{et}} = X_{\text{Nis}}$. The Leray spectral sequence together with the fact that the Nisnevich cohomological dimension of noetherian schemes is bounded by the Krull dimension [Nis, Thm. 1.32] yields:

Corollary 3.3. *Let X be a noetherian scheme of finite Krull dimension d , $T \subset X$ closed and assume that there exists a nonnegative integer N such that*

$$cd(k(x)) \leq N$$

for all points $x \in X \setminus T$. Then for every abelian torsion sheaf F on $(X, T)_{\text{et}}$ we have

$$H_{\text{et}}^p(X, T, F) = 0 \quad \text{for } p > N + d.$$

4 Fundamental group

We recall some facts from [AM]. Let \mathcal{C} be a pointed site and $\text{HR}(\mathcal{C})$ the category of pointed hypercoverings of \mathcal{C} [AM, §8]. If \mathcal{C} is locally connected, then the “connected component functor” π defines an object

$$\Pi \mathcal{C} = \{\pi(K_\bullet)\}_{K_\bullet \in \text{HR}(\mathcal{C})}$$

in the pro-category of the homotopy category of pointed simplicial sets. By definition, the fundamental group of \mathcal{C} is the pro-group $\pi_1(\Pi(\mathcal{C}))$.

Let X be a locally noetherian scheme. Then (cf. [AM, §9]) the site $(X, T)_{\text{et}}$ is locally connected. Pointing $(X, T)_{\text{et}}$ (e.g. by any of the points described at the end of section 1), we obtain the étale fundamental group $\pi_1^{\text{et}}(X, T)$. By [AM, Cor. 10.7], for any group G , the set $\text{Hom}(\pi_1^{\text{et}}(X, T), G)$ is in bijection with the set of

isomorphism classes of pointed G -torsors in $(X, T)_{\text{et}}$. In particular, $\pi_1^{\text{et}}(X, \emptyset)$ is the enlarged étale fundamental group of [SGA3, X, §6] and its profinite completion the usual étale fundamental group of X defined in [SGA1]. Since $\pi_1^{\text{et}}(X, T)$ is a factor group of $\pi_1^{\text{et}}(X, \emptyset)$, which is profinite for normal X by [AM, Thm. 11.1], we obtain the following result.

Proposition 4.1. *Let X be a noetherian, normal connected scheme and $T \subset X$. Then $\pi_1^{\text{et}}(X, T)$ is profinite. Its finite quotients are in bijection with the isomorphism classes of finite connected étale Galois coverings of X in which every point $t \in T$ splits completely.*

Example 4.2. For general (X, T) , the fundamental group need not be profinite. For example, let k be a field and $N = \mathbb{P}_k^1/(0 \sim 1)$ the node over k . Then

$$\pi_1^{\text{et}}(N, T) \cong \begin{cases} \mathbb{Z} \times G_k, & T = \emptyset \\ \mathbb{Z}, & T = X. \end{cases}$$

5 A modification

We consider a modification of the marked étale site which was used in [Scm] for one-dimensional noetherian regular schemes.

Definition 5.1. The *strict marked étale site* $(X, T)_{\text{et-s}}$ consists of the following data: $\text{Cat}(X, T)_{\text{et-s}}$ is the category of morphisms $f : (U, S) \rightarrow (X, T)$ such that

- a) $(f : U \rightarrow X)$ is a separated étale morphism of finite presentation,
- b) $S = p^{-1}(T)$, and
- c) for every $u \in S$ mapping to $t \in T$ the induced field homomorphism $k(s) \rightarrow k(u)$ is an isomorphism.

Coverings are surjective families.

Proposition 5.2. (i) *If $T \subset X$ consists of a finite set of closed points, then the natural morphism of sites $\varphi : (X, T)_{\text{et}} \rightarrow (X, T)_{\text{et-s}}$ induces isomorphisms*

$$H_{\text{et-s}}^p(X, T, F) \xrightarrow{\sim} H_{\text{et}}^p(X, T, \varphi^* F), \quad H_{\text{et-s}}^p(X, T, \varphi_* G) \xrightarrow{\sim} H_{\text{et}}^p(X, T, G)$$

for any $F \in \text{Sh}_{\text{et-s}}(X, T)$, $G \in \text{Sh}_{\text{et-s}}(X, T)$ and $p \geq 0$.

(ii) *For locally noetherian X (and any chosen base point)*

$$\pi_1^{\text{et}}(X, T) \xrightarrow{\sim} \pi_1^{\text{et-s}}(X, T).$$

Proof. Let $(U, S) \in \text{Cat}(X, T)_{\text{et-s}}$ and assume that $(f_i : (U_i, S_i) \rightarrow (U, S))$ is a covering in $(X, T)_{\text{et}}$. Removing for all i the finitely many points $s \in S_i$ such that $k(f(s_i)) \rightarrow k(s_i)$ is not an isomorphism from U_i , we obtain a strict covering $(f_i : (U'_i, S'_i) \rightarrow (U, S))$ which is a refinement of the original one. Hence $\varphi_* \varphi^* F = F$ and $R^q \varphi_* G = 0$ for $q \geq 1$. In view of the Leray spectral sequence, this shows (i). Assertion (ii) follows since both pro-groups represent the same functor: for any group G , a G -torsor in $(X, T)_{\text{et}}$ is the same as a G -torsor in $(X, T)_{\text{et-s}}$. \square

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